

# Toy models for D. H. Lehmer's conjecture II

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## Abstract.

In the previous paper, we studied the “Toy models for D. H. Lehmer's conjecture”. Namely, we showed that the  $m$ -th Fourier coefficient of the weighted theta series of the  $\mathbb{Z}^2$ -lattice and the  $A_2$ -lattice does not vanish, when the shell of norm  $m$  of those lattices is not the empty set. In other words, the spherical 4 (resp. 6)-design does not exist among the nonempty shells in the  $\mathbb{Z}^2$ -lattice (resp.  $A_2$ -lattice).

This paper is the sequel to the previous paper. We take 2-dimensional lattices associated to the algebraic integers of imaginary quadratic fields whose class number is either 1 or 2, except for  $\mathbb{Q}(\sqrt{-1})$  and  $\mathbb{Q}(\sqrt{-3})$ , then, show that the  $m$ -th Fourier coefficient of the weighted theta series of those lattices does not vanish, when the shell of norm  $m$  of those lattices is not the empty set. Equivalently, we show that the corresponding spherical 2-design does not exist among the nonempty shells in those lattices.

**Key Words and Phrases.** weighted theta series, spherical  $t$ -design, modular forms, lattices, Hecke operator.

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# 1 Introduction

The concept of spherical  $t$ -design is due to Delsarte-Goethals-Seidel [7]. For a positive integer  $t$ , a finite nonempty subset  $X$  of the unit sphere

$$S^{n-1} = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1^2 + x_2^2 + \dots + x_n^2 = 1\}$$

is called a spherical  $t$ -design on  $S^{n-1}$  if the following condition is satisfied:

$$\frac{1}{|X|} \sum_{x \in X} f(x) = \frac{1}{|S^{n-1}|} \int_{S^{n-1}} f(x) d\sigma(x),$$

for all polynomials  $f(x) = f(x_1, x_2, \dots, x_n)$  of degree not exceeding  $t$ . Here, the righthand side means the surface integral on the sphere, and  $|S^{n-1}|$  denotes the surface volume of the sphere  $S^{n-1}$ . The meaning of spherical  $t$ -design is that the average value of the integral of any polynomial of degree up to  $t$  on the sphere is replaced by the average value at a finite set on the sphere. A finite subset  $X$  in  $S^{n-1}(r)$ , the sphere of radius  $r$  centered at the origin, is also called a spherical  $t$ -design if  $\frac{1}{r}X$  is a spherical  $t$ -design on the unit sphere  $S^{n-1}$ .

We denote by  $\text{Harm}_j(\mathbb{R}^n)$  the set of homogeneous harmonic polynomials of degree  $j$  on  $\mathbb{R}^n$ . It is well known that  $X$  is a spherical  $t$ -design if and only if the condition

$$\sum_{x \in X} P(x) = 0$$

holds for all  $P \in \text{Harm}_j(\mathbb{R}^n)$  with  $1 \leq j \leq t$ . If the set  $X$  is antipodal, that is  $-X = X$ , and  $j$  is odd, then the above condition is fulfilled automatically. So we reformulate the condition of spherical  $t$ -design on the antipodal set as follows:

**Proposition 1.1.** *A nonempty finite antipodal subset  $X \subset S^{n-1}$  is a spherical  $2s+1$ -design if the condition*

$$\sum_{x \in X} P(x) = 0$$

*holds for all  $P \in \text{Harm}_{2j}(\mathbb{R}^n)$  with  $2 \leq 2j \leq 2s$ .*

It is known [7] that there is a natural lower bound (Fisher type inequality) for the size of a spherical  $t$ -design in  $S^{n-1}$ . Namely, if  $X$  is a spherical  $t$ -design in  $S^{n-1}$ , then

$$|X| \geq \binom{n-1+[t/2]}{[t/2]} + \binom{n+[t/2]-2}{[t/2]-1}$$

if  $t$  is even, and

$$(1) \quad |X| \geq 2 \binom{n-1+[t/2]}{[t/2]}$$

if  $t$  is odd.

A lattice in  $\mathbb{R}^n$  is a subset  $\Lambda \subset \mathbb{R}^n$  with the property that there exists a basis  $\{v_1, \dots, v_n\}$  of  $\mathbb{R}^n$  such that  $\Lambda = \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_n$ , i.e.,  $\Lambda$  consists of all integral linear combinations of the vectors  $v_1, \dots, v_n$ . The dual lattice  $\Lambda^\sharp$  is the lattice

$$\Lambda^\sharp := \{y \in \mathbb{R}^n \mid (y, x) \in \mathbb{Z}, \text{ for all } x \in \Lambda\},$$

where  $(x, y)$  is the standard Euclidean inner product. The lattice  $\Lambda$  is called integral if  $(x, y) \in \mathbb{Z}$  for all  $x, y \in \Lambda$ . An integral lattice is called even if  $(x, x) \in 2\mathbb{Z}$  for all  $x \in \Lambda$ , and it is odd otherwise. An integral lattice is called unimodular if  $\Lambda^\sharp = \Lambda$ . For a lattice  $\Lambda$  and a positive real number  $m > 0$ , the shell of norm  $m$  of  $\Lambda$  is defined by

$$\Lambda_m := \{x \in \Lambda \mid (x, x) = m\} = \Lambda \cap S^{n-1}(m).$$

Let  $\mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$  be the upper half-plane.

**Definition 1.1.** Let  $\Lambda$  be the lattice of  $\mathbb{R}^n$ . Then for a polynomial  $P$ , the function

$$\Theta_{\Lambda, P}(z) := \sum_{x \in \Lambda} P(x) e^{i\pi z(x, x)}$$

is called the theta series of  $\Lambda$  weighted by  $P$ .

**Remark 1.1** (See Hecke [8], Schoeneberg [18, 19]).

(i) When  $P = 1$ , we get the classical theta series

$$\Theta_\Lambda(z) = \Theta_{\Lambda, 1}(z) = \sum_{m \geq 0} |\Lambda_m| q^m, \text{ where } q = e^{\pi i z}.$$

(ii) The weighted theta series can be written as

$$\begin{aligned}\Theta_{\Lambda, P}(z) &= \sum_{x \in \Lambda} P(x) e^{i\pi z(x, x)} \\ &= \sum_{m \geq 0} a_m^{(P)} q^m, \text{ where } a_m^{(P)} := \sum_{x \in \Lambda_m} P(x).\end{aligned}$$

These weighted theta series have been used efficiently for the study of spherical designs which are the nonempty shells of Euclidean lattices. (See [22, 23, 5, 15, 6]. See also [2].)

**Lemma 1.1** (cf. [22, 23], [15, Lemma 5]). *Let  $\Lambda$  be an integral lattice in  $\mathbb{R}^n$ . Then, for  $m > 0$ , the non-empty shell  $\Lambda_m$  is a spherical  $t$ -design if and only if*

$$a_m^{(P)} = 0$$

for all  $P \in \text{Harm}_{2j}(\mathbb{R}^n)$  with  $1 \leq 2j \leq t$ , where  $a_m^{(P)}$  are the Fourier coefficients of the weighted theta series

$$\Theta_{\Lambda, P}(z) = \sum_{m \geq 0} a_m^{(P)} q^m.$$

The theta series of  $\Lambda$  weighted by  $P$  is a modular form for some subgroup of  $SL_2(\mathbb{R})$ . We recall the definition of the modular forms.

**Definition 1.2.** Let  $\Gamma \subset SL_2(\mathbb{R})$  be a Fuchsian group of the first kind and let  $\chi$  be a character of  $\Gamma$ . A holomorphic function  $f : \mathbb{H} \rightarrow \mathbb{C}$  is called a modular form of weight  $k$  for  $\Gamma$  with respect to  $\chi$ , if the following conditions are satisfied:

- (i)  $f\left(\frac{az+b}{cz+d}\right) = \left(\frac{cz+d}{\chi(\sigma)}\right)^k f(z)$  for all  $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ .
- (ii)  $f(z)$  is holomorphic at every cusp of  $\Gamma$ .

If  $f(z)$  has period  $N$ , then  $f(z)$  has a Fourier expansion at infinity, [10]:

$$f(z) = \sum_{m=0}^{\infty} a_m q_N^m, \quad q_N = e^{2\pi iz/N}.$$

We remark that for  $m < 0$ ,  $a_m = 0$ , by the condition (ii). A modular form with constant term  $a_0 = 0$ , is called a cusp form. We denote by  $M_k(\Gamma, \chi)$  (resp.  $S_k(\Gamma, \chi)$ ) the space of modular forms (resp. cusp forms) with respect to  $\Gamma$  with the character  $\chi$ . When  $f$  is the normalized eigenform of Hecke operators, p.163, [10], the Fourier coefficients satisfy the following relations:

**Lemma 1.2** (cf. [10, Proposition 32, 37, 40, Exercise 2, p.164]). *Let  $f(z) = \sum_{m \geq 1} a(m)q^m \in S_k(\Gamma, \chi)$ . If  $f(z)$  is the normalized eigenform of Hecke operators, then the Fourier coefficients of  $f(z)$  satisfy the following relations:*

$$\begin{aligned} (2) \quad a(mn) &= a(m)a(n) \text{ if } (m, n) = 1 \\ (3) \quad a(p^{\alpha+1}) &= a(p)a(p^\alpha) - \chi(p)p^{k-1}a(p^{\alpha-1}) \text{ if } p \text{ is a prime.} \end{aligned}$$

We set  $f(z) = \sum_{m \geq 1} a(m)q^m \in S_k(\Gamma, \chi)$ . When  $\dim S_k(\Gamma, \chi) = 1$  and  $a(1) = 1$ , then  $f(z)$  is the normalized eigenform of Hecke operators, [10]. So, the coefficients of  $f(z)$  have the relations as mentioned in Lemma 1.2. It is known that

$$(4) \quad |a(p)| < 2p^{(k-1)/2}$$

for all primes  $p$ . Note that this is the Ramanujan conjecture and its generalization, called the Ramanujan-Petersson conjecture for cusp forms which are eigenforms of the Hecke operators. These conjectures were proved by Deligne as a consequence of his proof of the Weil conjectures, [10, page 164], [9]. Moreover, for a prime  $p$  with  $\chi(p) = 1$  the following equation holds, [11].

$$(5) \quad a(p^\alpha) = p^{(k-1)\alpha/2} \frac{\sin(\alpha+1)\theta_p}{\sin \theta_p},$$

where  $2 \cos \theta_p = a(p)p^{-(k-1)/2}$ .

It is well known that the theta series of  $\Lambda \subset \mathbb{R}^n$  weighted by harmonic polynomial  $P \in \text{Harm}_j(\mathbb{R}^n)$  is a modular form of weight  $n/2 + j$  for some subgroup  $\Gamma \subset SL_2(\mathbb{R})$ . In particular, when  $\deg(P) \geq 1$ , the theta series of  $\Lambda$  weighted by  $P$  is a cusp form.

For example, we consider the even unimodular lattice  $\Lambda$ . Then the theta series of  $\Lambda$  weighted by harmonic polynomial  $P$ ,  $\Theta_{\Lambda, P}(z)$ , is a modular form with respect to  $SL_2(\mathbb{Z})$ .

**Example 1.1.** Let  $\Lambda$  be the  $E_8$ -lattice. This is an even unimodular lattice of  $\mathbb{R}^8$ , generated by the  $E_8$  root system. The theta series is as follows:

$$\begin{aligned}\Theta_\Lambda(z) = E_4(z) &= 1 + 240 \sum_{m=1}^{\infty} \sigma_3(m) q^{2m} \\ &= 1 + 240q^2 + 2160q^4 + 6720q^6 + 17520q^8 + \cdots,\end{aligned}$$

where  $\sigma_3(m)$  is a divisor function  $\sigma_3(m) = \sum_{0 < d|m} d^3$ .

For  $j = 2, 4$  and  $6$ , the theta series of  $\Lambda$  weighted by  $P \in \text{Harm}_j(\mathbb{R}^8)$  is a weight  $6, 8$  and  $10$  cusp form with respect to  $SL_2(\mathbb{Z})$ . However, it is well known that for  $k = 6, 8$  and  $10$ ,  $\dim S_k(SL_2(\mathbb{Z})) = 0$ , that is,  $\Theta_{\Lambda, P}(z) = 0$ . Then by Lemma 1.1, all the nonempty shells of  $E_8$ -lattice are spherical 6-design.

For  $j = 8$ , the theta series of  $\Lambda$  weighted by  $P$  is a weight  $12$  cusp form with respect to  $SL_2(\mathbb{Z})$ . Such a cusp form is uniquely determined up to constant, i.e., it is Ramanujan's delta function:

$$\Delta_{24}(z) = q^2 \prod_{m \geq 1} (1 - q^{2m})^{24} = \sum_{m \geq 1} \tau(m) q^{2m}.$$

The following proposition is due to Venkov, de la Harpe and Pache [5, 6, 15, 22].

**Proposition 1.2** (cf. [15]). *Let the notation be the same as above. Then the following are equivalent:*

- (i)  $\tau(m) = 0$ .
- (ii)  $(\Lambda)_{2m}$  is an 8-design.

It is a famous conjecture of Lehmer that  $\tau(m) \neq 0$ . So, Proposition 1.2 gives a reformulation of Lehmer's conjecture. Lehmer proved in [11] the following theorem.

**Theorem 1.1** (cf. [11]). *Let  $m_0$  be the least value of  $m$  for which  $\tau(m) = 0$ . Then  $m_0$  is a prime if it is finite.*

There are many attempts to study Lehmer's conjecture ([11, 20]), but it is difficult to prove and it is still open.

Recently, however, we showed the “Toy models for D. H. Lehmer’s conjecture” [3]. We take the two cases  $\mathbb{Z}^2$ -lattice and  $A_2$ -lattice. Then, we consider the analogue of Lehmer’s conjecture corresponding to the theta series weighted by some harmonic polynomial  $P$ . Namely, we show that the  $m$ -th coefficient of the weighted theta series of  $\mathbb{Z}^2$ -lattice does not vanish when the shell of norm  $m$  of those lattices is not an empty set. Or equivalently, we show the following result.

**Theorem 1.2** (cf. [3]). *The nonempty shells in  $\mathbb{Z}^2$ -lattice (resp.  $A_2$ -lattice) are not spherical 4-designs (resp. 6-designs).*

This paper is sequel to the previous paper [3]. In this paper, we take some lattices related to the imaginary quadratic fields. Let  $K = \mathbb{Q}(\sqrt{-d})$  be an imaginary quadratic field, and let  $\mathcal{O}_K$  be its ring of algebraic integers. Let  $\text{Cl}_K$  be the ideal classes. In this paper, we only consider the cases  $|\text{Cl}_K| = 1$  and  $|\text{Cl}_K| = 2$  except for Section 6. So, when we consider the cases  $|\text{Cl}_K| = 1$  and  $|\text{Cl}_K| = 2$ , we denote by  $\mathfrak{o}$  (resp.  $\mathfrak{a}$ ) the principal (resp. nonprincipal) ideal class.

We denote by  $d_K$  the discriminant of  $K$ :

$$d_K = \begin{cases} -4d & \text{if } -d \equiv 2, 3 \pmod{4}, \\ -d & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

**Theorem 1.3** (cf. [24, page 87]). *Let  $d$  be a positive square-free integer, and let  $K = \mathbb{Q}(\sqrt{-d})$ . Then*

$$\mathcal{O}_K = \begin{cases} \mathbb{Z} + \mathbb{Z}\sqrt{-d} & \text{if } -d \equiv 2, 3 \pmod{4}, \\ \mathbb{Z} + \mathbb{Z}\frac{-1 + \sqrt{-d}}{2} & \text{if } -d \equiv 1 \pmod{4}. \end{cases}$$

Therefore, we consider  $\mathcal{O}_K$  to be the lattice in  $\mathbb{R}^2$  with the basis

$$\begin{cases} (1, 0), (1, \sqrt{-d}) & \text{if } -d \equiv 2, 3 \pmod{4}, \\ (1, 0), \left(-\frac{1}{2}, \frac{\sqrt{-d}}{2}\right) & \text{if } -d \equiv 1 \pmod{4}, \end{cases}$$

denoted by  $L_{\mathfrak{o}}$ .

Generally, it is well-known that there exists one-to-one correspondence between the set of reduced quadratic forms  $f(x, y)$  with a fundamental discriminant  $d_K < 0$  and the set of fractional ideal classes of the unique quadratic

field  $\mathbb{Q}(\sqrt{-d})$  [24, page 94]. Namely, For a fractional ideal  $A = \mathbb{Z}\alpha + \mathbb{Z}\beta$ , we obtain the quadratic form  $ax^2 + bxy + cy^2$ , where  $a = \alpha\alpha'/N(A)$ ,  $b = (\alpha\beta' + \alpha'\beta)/N(A)$  and  $c = \beta\beta'/N(A)$ . Conversely, for a quadratic form  $ax^2 + bxy + cy^2$ , we obtain the fractional ideal  $\mathbb{Z} + \mathbb{Z}(b + \sqrt{d_K})/2a$ . We remark that  $N(A)$  is a norm of  $A$  and  $\alpha'$  is a complex conjugate of  $\alpha$ .

Here, we define the automorphism group of  $f(x, y)$  as follows:

$$U_f = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z}) \mid f(\alpha x + \beta y, \gamma x + \delta y) = f(x, y) \right\}.$$

Then, for  $n \geq 1$ , the number of the nonequivalent solutions of  $f(x, y) = n$  under the action of  $U_f$  is equal to the number of the integral ideals of norm  $n$  [24].

**Theorem 1.4** (cf. [24, page 63]). *Let  $f(x, y)$  be the reduced quadratic form with a fundamental discriminant  $D < 0$  and  $U_f$  be the automorphism group of  $f(x, y)$ . Then*

$$\#U_f = \begin{cases} 6 & \text{if } D = -3, \\ 4 & \text{if } D = -4, \\ 2 & \text{if } D < -4. \end{cases}$$

These classical results are due to Gauss, Dirichlet, etc. Let  $\mathfrak{a}$  be an ideal class and  $f_{\mathfrak{a}}(x, y)$  be the reduced quadratic form corresponding to  $\mathfrak{a}$ . Moreover, let  $L_{\mathfrak{a}}$  be the lattice corresponding to  $f(x, y)$ . We denote by  $N(A)$  the norm of an ideal  $A$ . Then, using Theorem 1.4, we have

$$(6) \quad \sum_{x \in L_{\mathfrak{a}}} q^{(x, x)} = 1 + \#U_f \sum_{n=1}^{\infty} \#\{A \mid A \text{ is an integral ideal of } \mathfrak{a}, N(A) = n\} q^n.$$

When  $|\text{Cl}_K| = 2$ , we give the generators of  $L_{\mathfrak{a}}$  in Appendix. Here, we remark that when  $K = \mathbb{Q}(\sqrt{-1})$  (resp.  $K = \mathbb{Q}(\sqrt{-3})$ ),  $L_{\mathfrak{o}}$  is  $\mathbb{Z}^2$ -lattice (resp.  $A_2$ -lattice). We studied the spherical designs of shells of those lattices in the previous paper [3].

In this paper, we take the imaginary quadratic fields  $\mathbb{Q}(\sqrt{-d})$ , with  $d \neq 1$  and  $d \neq 3$ . Then, we consider the analogue of Lehmer's conjecture corresponding to its theta series weighted by some harmonic polynomial  $P$ . Here, we consider the following problem that whether the nonempty shells of  $L_{\mathfrak{o}}$  and  $L_{\mathfrak{a}}$  are spherical 2-designs (hence 3-designs) or not.



In Section 4, we study the case that the class number is 1. We show that the  $m$ -th coefficient of the weighted theta series of  $L_{\mathfrak{o}}$ -lattice does not vanish when the shell of norm  $m$  of those lattices is not an empty set. Or equivalently, we show the following result:

**Theorem 1.5.** *Let  $K = \mathbb{Q}(\sqrt{-d})$  be an imaginary quadratic field whose class number is 1 and  $d \neq 1, 3$  i.e.,  $d$  is in the following set:  $\{2, 7, 11, 19, 43, 67, 163\}$ . Then, the nonempty shells in  $L_{\mathfrak{o}}$  are not spherical 2-designs.*

Similarly, in Section 5, we study the case that the class number is 2 and show the following result:

**Theorem 1.6.** *Let  $K = \mathbb{Q}(\sqrt{-d})$  be an imaginary quadratic field whose class number is 2 i.e.,  $d$  is in the following set:  $\{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}$ . Then, the nonempty shells in  $L_{\mathfrak{o}}$  and  $L_{\mathfrak{a}}$  are not spherical 2-designs.*

In Section 6, we consider the case that the class number is 3 and study the property of Hecke characters. In Section 7, we give some concluding remarks and state a conjecture for the future study.

## 2 Preliminaries

In this section, we review the theory of imaginary quadratic fields.

**Theorem 2.1** (cf. [4, page 104, Proposition 5.16]). *We can classify the prime ideals of a quadratic field as follows:*

1. *If  $p$  is an odd prime and  $(d_K/p) = 1$  (resp.  $d_K \equiv 1 \pmod{8}$ ) then  $(p) = P\bar{P}$  (resp.  $(2) = P\bar{P}$ ), where  $P$  and  $\bar{P}$  are prime ideals with  $P \neq \bar{P}$ ,  $N(P) = N(\bar{P}) = p$  (resp.  $N(P) = 2$ ).*
2. *If  $p$  is an odd prime and  $(d_K/p) = -1$  (resp.  $d_K \equiv 5 \pmod{8}$ ) then  $(p) = P$  (resp.  $(2) = P$ ), where  $P$  is a prime ideal with  $N(P) = p^2$  (resp.  $N(P) = 4$ ).*
3. *If  $p \mid d_K$  then  $(p) = P^2$ , where  $P$  is a prime ideal with  $N(P) = p$ .*

**Lemma 2.1.** *Let  $I$  be an integral ideal of  $K$ . For  $n \in \mathbb{N}$ , if  $N(I) = n$  and  $I$  is a principal ideal, namely,  $I \in \mathfrak{o}$  then there exist  $a, b \in \mathbb{Z}$  such that for  $-d \equiv 2, 3 \pmod{4}$*

$$n = a^2 + db^2,$$

for  $-d \equiv 1 \pmod{4}$

$$n = a^2 + db^2 \quad \text{or} \quad n = \frac{a^2 + db^2}{4}.$$

*If  $|\text{Cl}_K| = 2$ ,  $N(I) = n$  and  $I$  is a nonprincipal ideal, namely,  $I \in \mathfrak{a}$  and assume that  $m$  is one of the norm of nonprincipal ideals then there exist  $a, b \in \mathbb{Z}$  such that for  $-d \equiv 2, 3 \pmod{4}$*

$$mn = a^2 + db^2,$$

for  $-d \equiv 1 \pmod{4}$

$$mn = a^2 + db^2 \quad \text{or} \quad mn = \frac{a^2 + db^2}{4}.$$

*Proof.* We assume that  $|\text{Cl}_K| = 1$ . For  $-d \equiv 2, 3 \pmod{4}$ , we can write  $I = (a + b\sqrt{-d})$ , then  $N(I) = a^2 + db^2$ . For  $-d \equiv 1 \pmod{4}$ , we can write  $I = (a + b\sqrt{-d})$  or  $I = ((a + b\sqrt{-d})/2)$ , then  $N(I) = a^2 + db^2$  or  $N(I) = (a^2 + db^2)/4$ .

Here, we assume that  $|\text{Cl}_K| = 2$ . Let  $J$  be the nonprincipal ideal of  $K$  whose norm is  $m$ . If  $I$  is a nonprincipal ideal then,  $JI$  is a principal ideal of  $K$ . Therefore, for  $-d \equiv 2, 3 \pmod{4}$ , we can write  $JI = (a + b\sqrt{-d})$ , then  $N(JI) = a^2 + db^2$ . Hence,  $mn = a^2 + db^2$ . for  $-d \equiv 1 \pmod{4}$ , we can write  $JI = (a + b\sqrt{-d})$  or  $JI = ((a + b\sqrt{-d})/2)$ , then  $N(JI) = a^2 + db^2$  or  $N(JI) = (a^2 + db^2)/4$ . Hence,  $mn = a^2 + db^2$  or  $mn = (a^2 + db^2)/4$ .  $\square$

**Proposition 2.1.** *Let  $F(m)$  be the number of the integral ideals of norm  $m$  of  $K$ . Let  $p$  be a prime number. Then, if  $p \neq 2$*

$$F(p^e) = \begin{cases} e + 1 & \text{if } (d_K/p) = 1, \\ (1 + (-1)^e)/2 & \text{if } (d_K/p) = -1, \\ 1 & \text{if } p \mid d_K, \end{cases}$$

if  $p = 2$

$$F(2^e) = \begin{cases} e + 1 & \text{if } d_K \equiv 1 \pmod{8}, \\ (1 + (-1)^e)/2 & \text{if } d_K \equiv 5 \pmod{8}, \\ 1 & \text{if } 2 \mid d_K. \end{cases}$$

*Proof.* When  $(d_K/p) = 1$  i.e.,  $(p) = P\overline{P}$  and  $P \neq \overline{P}$ , since  $P$  and  $\overline{P}$  are the only integral ideals of norm  $p$ , we have  $F(p) = 2$ . Moreover, the integral ideals of norm  $p^e$  are as follows:  $P^e, P^{e-1}\overline{P}, \dots, (\overline{P})^e$ . So, we have  $F(p^e) = e + 1$ . The other cases can be proved similarly.  $\square$

### 3 Hecke characters and Theta series

In this section, we introduce the Hecke character and discuss the relationships between the Hecke character and the weighted theta series of the lattices  $L_{\mathfrak{o}}$  and  $L_{\mathfrak{a}}$ . Then, we show that for  $|\text{Cl}_K| = 1$  and  $P_1 = (x^2 - y^2)/2$ , the weighted theta series  $\Theta_{L_{\mathfrak{o}}, P_1}$  is a normalized Hecke eigenform. For  $|\text{Cl}_K| = 2$  and  $P_2 = x^2 - y^2$ , a certain sum of the two weighted theta series  $c_1\Theta_{L_{\mathfrak{o}}, P_2} + c_2\Theta_{L_{\mathfrak{a}}, P_2}$  is a normalized Hecke eigenform. Later, we give the explicit values of  $c_1$  and  $c_2$ .

A Hecke character  $\phi$  of weight  $k \geq 2$  with modulus  $\Lambda$  is defined in the following way. Let  $\Lambda$  be a nontrivial ideal in  $\mathcal{O}_K$  and let  $I(\Lambda)$  denote the group of fractional ideals prime to  $\Lambda$ . A Hecke character  $\phi$  with modulus  $\Lambda$  is a homomorphism

$$\phi : I(\Lambda) \rightarrow \mathbb{C}^\times$$

such that for each  $\alpha \in K^\times$  with  $\alpha \equiv 1 \pmod{\Lambda}$  we have

$$(7) \quad \phi(\alpha\mathcal{O}_K) = \alpha^{k-1}.$$

Let  $\omega_\phi$  be the Dirichlet character with the property that

$$\omega_\phi(n) := \phi((n))/n^{k-1}$$

for every integer  $n$  coprime to  $\Lambda$ .

**Theorem 3.1** (cf. [14, page 9], [13, page 183]). *Let the notation be the same as above, and define  $\Psi_{K,\Lambda}(z)$  by*

$$(8) \quad \Psi_{K,\Lambda}(z) := \sum_A \phi(A)q^{N(A)} = \sum_{n=1}^{\infty} a(n)q^n,$$

where the sum is over the integral ideals  $A$  that are prime to  $\Lambda$  and  $N(A)$  is the norm of the ideal  $A$ . Then  $\Psi_{K,\Lambda}(z)$  is a cusp form in  $S_k(\Gamma_0(d_K \cdot N(\Lambda)), \left(\frac{-d_K}{\bullet}\right) \omega_\phi)$ .

We remark that function (8) is a normalized Hecke eigenform [1, 21]. Moreover, if the class number of  $K$  is  $h$  then the character as given in (7) will have  $h$  extensions to nonprincipal ideals. Namely, the function (8) has  $h$  choices, so we denote by  $\Psi_{K,\Lambda}^{(1)}(z), \dots, \Psi_{K,\Lambda}^{(h)}(z)$  each functions (see [16]).

**Example 3.1.**

(i)  $d = 2$ .

We calculate  $\Psi_{K,\Lambda}(z) = \sum_{m \geq 1} a(m)q^m$ , where  $\Lambda = (1)$  and the weight of the Hecke character is 3. We remark that  $|\text{Cl}_K| = 1$ . By the

Table 1: Integral ideals of small norm of  $d = 2$  and  $d = 5$

$N(A)$	$A$ : ideal	$N(A)$	$A$ : ideal
1	(1)	1	(1)
2	( $\sqrt{-2}$ )	2	( $2, 1 + \sqrt{-5}$ )
3	( $-1 + \sqrt{-2}$ ) ( $-1 - \sqrt{-2}$ )	3	( $3, 1 + \sqrt{-5}$ ) ( $3, 1 - \sqrt{-5}$ )
4	(2)	4	(2)
		5	( $\sqrt{-5}$ )
		6	( $1 - \sqrt{-5}$ ) ( $-1 - \sqrt{-5}$ )

definitions (7) and (8), we have  $a(1) = 1^2 = 1$ ,  $a(2) = \sqrt{-2}^2 = -2$ ,  $a(3) = (-1 + \sqrt{-2})^2 + (-1 - \sqrt{-2})^2 = 2$ ,  $a(4) = 2^2, \dots$ . Thus, we obtain

$$\Psi_{K,\Lambda}^{(1)}(z) = q - 2q^2 - 2q^3 + 4q^4 + 4q^6 - 8q^8 - 5q^9 + \dots$$

(ii)  $d = 5$ .

We calculate  $\Psi_{K,\Lambda}(z) = \sum_{m \geq 1} a(m)q^m$ , where  $\Lambda = (1)$  and the weight of the Hecke character is 3. We remark that  $|\text{Cl}_K| = 2$ . When  $A$  of norm  $m$  is a nonprincipal ideal,  $A^2$  is a principal ideal, so,  $\phi(A^2)$  is computable by the definition (7). For example,  $\phi((2, 1 + \sqrt{-5}))^2 = \phi((2)) = 4$ , so, we can assume that  $\phi((2, 1 + \sqrt{-5})) = 2$ , i.e.,  $a(2) = 2$ . Then, since  $(2, 1 + \sqrt{-5})(3, 1 + \sqrt{-5}) = (1 - \sqrt{-5})$  and  $(2, 1 + \sqrt{-5})(3, 1 - \sqrt{-5}) = (-1 - \sqrt{-5})$ , we have  $a(3) = ((1 + \sqrt{-5})^2 + (1 - \sqrt{-5})^2)/2 = -4$ ,  $a(4) = 2^2, \dots$ . Thus, we obtain

$$\Psi_{K,\Lambda}^{(1)}(z) = q + 2q^2 - 4q^3 + 4q^4 - 5q^5 - 8q^6 + 4q^7 + 8q^8 + 7q^9 + \dots$$

On the other hand, we assume that  $\phi((2, 1 + \sqrt{-5})) = -2$ , i.e.,  $a(2) = -2$ . Then, we have

$$\Psi_{K,\Lambda}^{(2)}(z) = q - 2q^2 + 4q^3 + 4q^4 - 5q^5 - 8q^6 - 4q^7 - 8q^8 + 7q^9 + \cdots.$$

Here, we discuss the relationships between the Hecke character and the weighted theta series of the lattices  $L_{\mathfrak{o}}$  and  $L_{\mathfrak{a}}$ . First, we quote the following theorem:

**Theorem 3.2** (cf. [13, page 192]). *Let  $L$  be an integral lattice with the Gram matrix  $A$  and  $N$  be the natural number such that the elements of  $NA^{-1}$  are rational integers. Let the character  $\chi(d)$  be*

$$\chi(d) = \left( \frac{(-1)^{(r/2)} \det L}{d} \right).$$

Then, for  $P \in \text{Harm}_2(\mathbb{R}^2)$ ,

- (1)  $\Theta_{L,P} \in M_3(\Gamma_0(4N), \chi)$ .
- (2) If all the diagonal elements of  $A$  are even, then  $\Theta_{L,P} \in M_3(\Gamma_0(2N), \chi)$ .
- (3) If all the diagonal elements of  $A$  and  $NA^{-1}$  are even, then  $\Theta_{L,P} \in M_3(\Gamma_0(N), \chi)$ .

Then, we obtain the following lemmas:

**Lemma 3.1.** *Let  $K$  be an imaginary quadratic field whose class number is 1 and  $L_{\mathfrak{o}}$  be the lattice corresponding to the principal ideal class  $\mathfrak{o}$ . Let  $\phi$  be the Hecke character of weight 3 with modulus  $\Lambda$ . Assume that  $\Lambda = (1)$  and  $P_1 = (x^2 - y^2)/2 \in \text{Harm}_2(\mathbb{R}^2)$ . Then,  $\Psi_{K,\Lambda}(q) = \Theta_{L_{\mathfrak{o}},P_1}(q)$ .*

**Lemma 3.2.** *Let  $K$  be an imaginary quadratic field whose class number is 2 and  $L_{\mathfrak{o}}$  (resp.  $L_{\mathfrak{a}}$ ) be the lattice corresponding to the principal ideal class  $\mathfrak{o}$  (resp. nonprincipal ideal class  $\mathfrak{a}$ ). Let  $\phi$  be the Hecke character of weight 3 with modulus  $\Lambda$ . Assume that  $\Lambda = (1)$  and  $P_2 = x^2 - y^2 \in \text{Harm}_2(\mathbb{R}^2)$ . Then,  $\Psi_{K,\Lambda}(q) = c_1 \Theta_{L_{\mathfrak{o}},P_2}(q) + c_2 \Theta_{L_{\mathfrak{a}},P_2}(q)$ , where  $c_1$  and  $c_2$  are given as in table 2.*

*Proof of Lemmas 3.1 and 3.2.* First, we assume that the lattices are integral lattices, if not we multiple the Gram matrix of  $L$  by 2.

Table 2: Coefficients,  $c_1$  and  $c_2$

$-d$	-5	-6	-10	-13	-15	-22	-35	-37	-51
$c_1$	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
$c_2$	1/2	1/2	1/2	1/2	2	1/2	3	1/2	1/2
$-d$	-58	-91	-115	-123	-187	-235	-267	-403	-427
$c_1$	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2	1/2
$c_2$	1/2	5/3	1/2	1/2	7/3	1/2	1/2	11/9	1/2

Because of the Theorems 3.1 and 3.2,  $\Psi_{K,\Lambda}(q)$ ,  $\Theta_{L_o,P}(q)$  and  $\Theta_{L_a,P}(q)$  with  $P = P_1, P_2$  are modular forms of the same group  $\Gamma$ . Therefore, we calculate the basis of the space of modular forms of group  $\Gamma$  and check  $\Psi_{K,\Lambda}(q) = \Theta_{L_o,P_1}(q)$  and  $\Psi_{K,\Lambda}(q) = c_1\Theta_{L_o,P_2}(q) + c_2\Theta_{L_a,P_2}(q)$  explicitly (using “Sage”, Mathematics Software [17]).  $\square$

**Corollary 3.1.** *Let the notation be the same as above. If  $|\text{Cl}_K| = 1$  then  $\Theta_{L_1,P_1}(q)$  is a normalized Hecke eigenform. If  $|\text{Cl}_K| = 2$  then  $c_1\Theta_{L_1,P_2}(q) + c_2\Theta_{L_2,P_2}(q)$  is a normalized Hecke eigenform.*

*Proof.* The function (8) is a normalized Hecke eigenform [1, 21].  $\square$

Finally, we give the following proposition, which is an analogue of Theorem 1.1 and the crucial part of the proof of Theorems 1.5 and 1.6.

**Proposition 3.1.** *Assume that  $\sum_{m \geq 1} a(m)q^m$  is a normalized Hecke eigenform of  $S_3(\Gamma, \chi)$  and the coefficients  $a(m)$  are rational integers. Moreover Let  $p$  be the prime such that  $\chi(p) = 1$ . Let  $\alpha_0$  be the least value of  $\alpha$  for which  $a(p^\alpha) = 0$ . If  $a(p) \neq \pm p$  then  $\alpha_0 = 1$  if it is finite.*

*Proof.* Assume the contrary, that is,  $\alpha_0 > 1$ , so that  $a(p) \neq 0$ . By the equation (5),

$$a(p^{\alpha_0}) = 0 = p^{\alpha_0} \frac{\sin(\alpha_0 + 1)\theta_p}{\sin \theta_p}.$$

This shows that  $\theta_p$  is a real number of the form  $\theta_p = \pi k / (1 + \alpha_0)$ , where  $k$  is an integer. Now the number

$$(9) \quad z = 2 \cos \theta_p = a(p)p^{-1},$$

being twice the cosine of a rational multiple of  $2\pi$ , is an algebraic integer. On the other hand  $z$  is a root of the equation

$$(10) \quad pz - a(p) = 0.$$

Hence  $z$  is a rational integer. By (4) and (9), we have  $|z| \leq 1$ . Therefore  $z = \pm 1$  and the equation (10) becomes  $a(p) = \pm p$ . By assumption, this is a contradiction.  $\square$

## 4 The case of $|\text{Cl}_K| = 1$

Let  $K := \mathbb{Q}(\sqrt{-d})$  be an imaginary quadratic field. If the class number of  $K$  is 1 then  $d$  is in the following set  $\{1, 2, 3, 7, 11, 19, 43, 67, 163\}$ . In particular, we only consider the cases where  $d$  is in the set:  $\{2, 7, 11, 19, 43, 67, 163\}$  since the cases  $d = 1$  and  $d = 3$  are considered in [3].

In this section, we assume that  $a(m)$  and  $b(m)$  are the coefficients of the following functions:

$$\Theta_{L_o}(q) = \sum_{m \geq 0} a(m)q^m, \quad \Theta_{L_o, P_1}(q) = \sum_{m \geq 1} b(m)q^m,$$

where  $P_1 = (x^2 - y^2)/2 \in \text{Harm}_2(\mathbb{R}^2)$ .

**Lemma 4.1.** *Let  $d$  be one of the elements in  $\{2, 7, 11, 19, 43, 67, 163\}$ . We set  $a'(m) = a(m)/2$  for all  $m$ . Then,*

$$a'(p^e) = \begin{cases} e + 1 & \text{if } (d_K/p) = 1, \\ (1 + (-1)^e)/2 & \text{if } (d_K/p) = -1, \\ 1 & \text{if } p \mid d_K. \end{cases}$$

*Proof.* Because of the equation (6),  $a'(m)$  is the number of integral ideals of  $K$  of norm  $m$ . Therefore, it can be proved by Proposition 2.1.  $\square$

**Lemma 4.2.** *Let  $p$  be a prime number such that  $(d_K/p) = 1$ . Then,  $b(p) \neq 0$ . Moreover, if  $p \neq d$  then  $b(p) \neq \pm p$ .*

*Proof.* We remark that by Corollary 3.1,  $\Theta_{L_o, P_1}(q) = \Psi_{K, \Lambda}(q)$ . So, the numbers  $b(m)$  are the coefficients of  $\Psi_{K, \Lambda}(q)$ .

First, we assume that  $d \neq 2$  i.e.,  $-d \equiv 1 \pmod{4}$  and  $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}(1 + \sqrt{-d})/2$ . If  $N((a + b\sqrt{-d}))$  is equal to  $p$  then by Lemma 2.1

$$p = a^2 + db^2.$$

Because of the definition of  $\Psi_{K,\Lambda}(q)$ ,

$$b(p) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).$$

If  $b(p) = 0$  then  $a^2 = db^2$ . This is a contradiction. Assume that  $b(p) = \pm p$ . Then,

$$2(a^2 - db^2) = \pm(a^2 + db^2),$$

that is,  $a^2 = 3db^2$  or  $3a^2 = db^2$ . This is a contradiction.

If  $N(((a + b\sqrt{-d})/2))$  is equal to  $p$  then by Lemma 2.1

$$\frac{a^2 + db^2}{4} = p.$$

Because of the definition of  $\Psi_{K,\Lambda}(q)$ ,

$$b(p) = \left(\frac{a + b\sqrt{-d}}{2}\right)^2 + \left(\frac{a - b\sqrt{-d}}{2}\right)^2 = \frac{a^2 - db^2}{2}.$$

If  $b(p) = 0$  then  $a^2 = db^2$ . This is a contradiction. Assume that  $b(p) = \pm p$ . Then,

$$\frac{a^2 - db^2}{2} = \pm \frac{a^2 + db^2}{4},$$

that is,  $a^2 = 3db^2$  or  $3a^2 = db^2$ . This is a contradiction.

Next, we assume that  $d = 2$  i.e.,  $-d \equiv 2 \pmod{4}$  and  $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$ . If  $N((a + b\sqrt{-2}))$  is equal to  $p$  then by Lemma 2.1

$$p = a^2 + 2b^2.$$

Because of the definition of  $\Psi_{K,\Lambda}(q)$ ,

$$b(p) = (a + b\sqrt{-2})^2 + (a - b\sqrt{-2})^2 = 2(a^2 - 2b^2).$$

If  $b(p) = 0$  then  $a^2 = 2b^2$ . This is a contradiction. Assume that  $b(p) = \pm p$ . Then,

$$2(a^2 - 2b^2) = \pm(a^2 + 2b^2),$$

that is,  $a^2 = 6b^2$  or  $3a^2 = 2b^2$ . This is a contradiction. □



*Proof of Theorem 1.5.* We will show that  $b(m) \neq 0$  when  $(L_{\mathfrak{o}})_m \neq \emptyset$ .

By Theorem 3.1,  $\Theta_{L_{\mathfrak{o}}, P_1}$  is a normalized Hecke eigenform. So, We assume that  $m$  is a power of prime, if not we could apply the equation (2). We will divide our considerations into the following three cases.

(i) Case  $m = p^\alpha$  and  $p \mid d_K$ :

By  $a(m) = 2$  and the inequality (1), the shells  $(L_{\mathfrak{o}})_m$  are not spherical 2-designs. Hence,  $b(m) \neq 0$ .

(ii) Case  $m = p^\alpha$  and  $(d_K/p) = -1$ :

By Lemma 4.1,

$$a(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

By  $a(m) = 2$  and the inequality (1), when  $n$  is even, the shells  $(L_{\mathfrak{o}})_m$  are not spherical 2-designs. Hence,  $b(m) \neq 0$ .

(iii) Case  $m = p^\alpha$  and  $(d_K/p) = 1$ :

By Proposition 3.1 and Lemma 4.2, we have  $b(m) \neq 0$ . This completes the proof of Theorem 1.5.  $\square$

## 5 The case of $|\text{Cl}_K| = 2$

Let  $K := \mathbb{Q}(\sqrt{-d})$  be an imaginary quadratic field. In this section, we assume that the class number of  $K$  is 2. So, we consider that  $d$  is in the following set:  $\{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}$ . We denote by  $\mathfrak{o}$  (resp.  $\mathfrak{a}$ ) the principal (resp. nonprincipal) ideal class.

In this section, we also assume that  $a(m)$  and  $b(m)$  are the coefficients of the following functions:

$$\begin{aligned} \Theta_{L_{\mathfrak{o}}}(q) + \Theta_{L_{\mathfrak{a}}}(q) &= \sum_{m \geq 0} a(m)q^m, \\ c_1 \Theta_{L_{\mathfrak{o}}, P_2}(q) + c_2 \Theta_{L_{\mathfrak{a}}, P_2}(q) &= \sum_{m \geq 1} b(m)q^m, \end{aligned}$$

where  $c_1$  and  $c_2$  are defined in Lemma 3.2.

**Lemma 5.1.** Set  $l_1 := \{N(O) \mid x \in L_o\}$  and  $l_2 := \{N(A) \mid A \in \mathfrak{a}\}$ . Then,  $l_1 \cap l_2 = \emptyset$ . Therefore, the set  $L_o \cap L_a$  consists of the origin.

*Proof.* Let  $p$  be the prime number such that  $(d_K/p) = 1$ . Then there exist prime ideals  $P$  and  $P'$  such that  $(p) = PP'$  and  $N(P) = N(P') = p$ . Since the class number is 2, we have  $P$  and  $P' \in \mathfrak{o}$  or  $P$  and  $P' \in \mathfrak{a}$ . If  $P$  and  $P' \in \mathfrak{o}$  we denote by  $p_i$  such a prime. If  $P$  and  $P' \in \mathfrak{a}$  we denote by  $p'_i$  such a prime.

Let  $p$  be the prime number such that  $(d_K/p) = -1$ . Then  $(p)$  is the prime ideal and  $N((p)) = p^2$ . We denote by  $q_i$  such a prime.

Let  $p$  be the prime number such that  $p \mid d_K$ . Then there exists a prime ideal  $P$  such that  $(p) = P^2$  and  $N(P) = p$ . Since the class number is 2, we have  $P \in \mathfrak{o}$  or  $P \in \mathfrak{a}$ . If  $P \in \mathfrak{o}$  we denote by  $r_i$  such a prime. If  $P \in \mathfrak{a}$  we denote by  $r'_i$  such a prime.

We take the element  $n \in l_1 \cap l_2$  and perform a prime factorization,  $n = p_1 \cdots p'_1 \cdots q_1 \cdots r_1 \cdots r'_1 \cdots$ . Then,  $p_1 \cdots$ ,  $q_1 \cdots$  and  $r_1 \cdots$  correspond to principal ideals. So, if the number of the primes  $p'$  and  $r'$  is even then  $n \in l_1$  and if the number of the primes  $p'$  and  $r'$  is odd then  $n \in l_2$ . This completes the proof of Lemma 5.1.  $\square$

**Lemma 5.2.** Let  $d$  be one of the elements in  $\{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}$ . We set  $a'(m) = a(m)/2$  for all  $m$ . Then,

$$a'(p^e) = \begin{cases} e + 1 & \text{if } (d_K/p) = 1, \\ (1 + (-1)^e)/2 & \text{if } (d_K/p) = -1, \\ 1 & \text{if } p \mid d_K. \end{cases}$$

*Proof.* Because of the equation (6),  $a'(m)$  is the number of integral ideals of  $K$  of norm  $m$ . Therefore, it can be proved by Proposition 2.1.  $\square$

**Lemma 5.3.** Let  $p$  be a prime number such that  $(d_K/p) = 1$ . Then,  $b(p) \neq 0$ . Moreover, if  $p \neq d$  then  $b(p) \neq \pm p$ .

*Proof.* We remark that by Corollary 3.1,  $c_1 \Theta_{L_o, P_2}(q) + c_2 \Theta_{L_a, P_2}(q) = \Psi_{K, \Lambda}(q)$ . So, the numbers  $b(m)$  are the coefficients of  $\Psi_{K, \Lambda}(q)$ .

We set  $N(J) = p$ . When  $J$  is a principal ideal, it can be proved by the similar method in Lemma 4.2. So, we assume that  $J$  is nonprincipal.

We list the smallest prime number  $m$  such that  $m \mid d_K$  and  $m \in \{N(I) \mid I \in \mathfrak{a}\}$ , and the values  $b(m)$  are in Table 3. First, we assume that  $-d \equiv 2$  or

Table 3: Values of  $m$  and  $b(m)$

$-d$	-5	-6	-10	-13	-15	-22	-35	-37	-51
$m$	2	2	2	2	3	2	5	2	3
$b(m)$	2	2	2	2	-3	2	-5	2	3
$-d$	-58	-91	-115	-123	-187	-235	-267	-403	-427
$m$	2	7	5	3	11	5	3	13	7
$b(m)$	2	-7	-5	3	-11	5	3	-13	7

3 (mod 4). If  $N(J)$  is equal to  $p$  then by Lemma 2.1

$$mp = a^2 + db^2.$$

Because of the definition of  $\Psi_{K,\Lambda}(q)$ ,

$$b(mp) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).$$

Since  $b(mp) = b(m)b(p)$  and the value of  $b(m)$  in Table 3, we have  $b(p) = a^2 - db^2$ . If  $b(p) = 0$  then  $a^2 = db^2$ . This is a contradiction. Assume that  $b(p) = \pm p$ . Then,

$$a^2 - db^2 = \pm \frac{a^2 + db^2}{2},$$

that is,  $a^2 = 3db^2$  or  $3a^2 = db^2$ . This is a contradiction.

Next, we assume that  $-d \equiv 1 \pmod{4}$ . If  $N(J)$  is equal to  $p$  then by Lemma 2.1 there exist  $a, b \in \mathbb{Z}$  such that

$$mp = a^2 + db^2 \quad \text{or} \quad mp = \frac{a^2 + db^2}{4}.$$

Because of the definition of  $\Psi_{K,\Lambda}(q)$ ,

$$b(mp) = (a + b\sqrt{-d})^2 + (a - b\sqrt{-d})^2 = 2(a^2 - db^2).$$

or

$$b(mp) = \left(\frac{a + b\sqrt{-d}}{2}\right)^2 + \left(\frac{a - b\sqrt{-d}}{2}\right)^2 = \frac{a^2 - db^2}{2}.$$

Since  $b(mp) = b(m)b(p)$  and the value of  $b(m)$  in Table 3, we have  $b(p) = 2/b(m) \times (a^2 - db^2)$  or  $b(p) = 1/b(m) \times (a^2 - db^2)/2$ . If  $b(p) = 0$  then  $a^2 = db^2$ . This is a contradiction. Assume that  $b(p) = \pm p$ . Then,

$$\frac{2(a^2 - db^2)}{b(m)} = \pm \frac{a^2 + db^2}{m},$$

or

$$\frac{a^2 - db^2}{2b(m)} = \pm \frac{a^2 + db^2}{4m},$$

that is,  $a^2 = 3db^2$  or  $3a^2 = db^2$  since  $m = \pm b(m)$  for  $-d \equiv 1 \pmod{4}$ . This is a contradiction.  $\square$

*Proof of Theorem 1.6.* Because of Lemma 5.1, it is enough to show that  $b(m) \neq 0$  when  $(L_o)_m \neq \emptyset$  or  $(L_a)_m \neq \emptyset$ .

By Theorem 3.1,  $c_1\Theta_{L_o, P_2} + c_2\Theta_{L_a, P_2}$  is a normalized Hecke eigenform. So, We assume that  $m$  is a power of prime, if not we could apply the equation (2). We will divide into the three cases.

- (i) Case  $m = p^\alpha$  and  $p \mid d_K$ :  
By  $a(m) = 2$  and (1), the shells  $(L)_m$  are not spherical 2-designs. Hence,  $b(m) \neq 0$ .
- (ii) Case  $m = p^\alpha$  and  $(d_K/p) = -1$ :  
By Lemma (4.1),

$$a(p^n) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

By  $a(m) = 2$  and (1), when  $n$  is even, the shells  $(L)_m$  are not spherical 2-designs. Hence,  $b(m) \neq 0$ .

- (iii) Case  $m = p^\alpha$  and  $(d_K/p) = 1$ :  
By Proposition 3.1 and Lemma 5.3,  $b(m) \neq 0$ . This completes the proof of Theorem 1.6.  $\square$

## 6 The case of $|\text{Cl}_K| = 3$

In the previous sections, we studied the cases of class number  $h = |\text{Cl}_K|$  is either 1 or 2. However, it seems that the situation is somewhat different for the cases of class numbers  $h \geq 3$ . In this section, we discuss briefly how it is different, by considering the case of  $d = 23$  ( $h = 3$ ).

We first remark that one reason of success for the cases  $h = 1$  and  $h = 2$  is that the coefficients  $a(m)$  of the Hecke eigenform  $\Psi_{K,\Lambda}$  are all integers. Therefore, by the formula (10)  $z = a(p)/p$  is a rational number (and since it is an algebraic integer), and so it must be a rational integer. It seems that this situation is no more true in general for the cases of  $h \geq 3$ . We will give more details information, concentrating the special (and typical) case of  $d = 23$ .

We denoted by  $\mathfrak{o}$ ,  $\mathfrak{a}_1$  and  $\mathfrak{a}_2$  the ideal classes. The corresponding quadratic forms are  $x^2 + xy + 6y^2$ ,  $2x^2 - xy + 3y^2$  and  $2x^2 + xy + 3y^2$ , namely,  $L_{\mathfrak{o}} = \langle (1, 0), (1/2, \sqrt{23}/2) \rangle$ ,  $L_{\mathfrak{a}_1} = \langle (2, 0), (1/2, \sqrt{23}/2) \rangle$  and  $L_{\mathfrak{a}_2} = \langle (2, 0), (-1/2, \sqrt{23}/2) \rangle$ , respectively. We give the weighed theta series of those ideal lattices. We set  $P_1 = x^2 - y^2$  and  $P_2 = xy$  in this section.

$$\Theta_{L_{\mathfrak{o}}} = 1 + 2q + 2q^4 + 4q^6 + 4q^8 + 2q^9 + 4q^{12} + 2q^{16} + 4q^{18} + 2q^{23} + 4q^{24} + 2q^{25} + 4q^{26} + 4q^{27} + 4q^{32} + 6q^{36} + 4q^{39} + 8q^{48} + 2q^{49} + 4q^{52} + 4q^{54} + 4q^{58} + 4q^{59} + 4q^{62} + 6q^{64} + 8q^{72} + 4q^{78} + 2q^{81} + 4q^{82} + 4q^{87} + 2q^{92} + 4q^{93} + 4q^{94} + 8q^{96} + 2q^{100} + O[q]^{101}$$

$$\frac{1}{2} \times \Theta_{L_{\mathfrak{o}}, P_1} = q + 4q^4 - 11q^6 - 7q^8 + 9q^9 + q^{12} + 16q^{16} + 13q^{18} - 23q^{23} - 44q^{24} + 25q^{25} + 29q^{26} - 38q^{27} - 28q^{32} + 85q^{36} - 14q^{39} + 77q^{48} + 49q^{49} - 103q^{52} - 99q^{54} - 91q^{58} + 26q^{59} + 101q^{62} - 15q^{64} - 11q^{72} + 133q^{78} + 81q^{81} - 43q^{82} + 82q^{87} - 92q^{92} - 182q^{93} - 19q^{94} - 7q^{96} + 100q^{100} + O[q]^{101}$$

$$\Theta_{L_{\mathfrak{o}}, P_2} = 0$$

$$\Theta_{L_{\mathfrak{a}_1}} = 1 + 2q^2 + 2q^3 + 2q^4 + 2q^6 + 2q^8 + 2q^9 + 4q^{12} + 2q^{13} + 4q^{16} + 4q^{18} + 6q^{24} + 2q^{26} + 2q^{27} + 2q^{29} + 2q^{31} + 4q^{32} + 6q^{36} + 2q^{39} + 2q^{41} + 2q^{46} + 2q^{47} + 6q^{48} + 2q^{50} + 4q^{52} + 6q^{54} + 2q^{58} + 2q^{62} + 4q^{64} + 2q^{69} + 2q^{71} + 8q^{72} + 2q^{73} + 2q^{75} + 6q^{78} + 4q^{81} + 2q^{82} + 2q^{87} + 2q^{92} + 2q^{93} + 2q^{94} + 8q^{96} + 2q^{98} + 2q^{100} + O[q]^{101}$$

$$2 \times \Theta_{L_{\mathfrak{a}_1}, P_1} = 8q^2 - 11q^3 - 7q^4 + q^6 + 32q^8 + 13q^9 - 88q^{12} + 29q^{13} - 56q^{16} + 121q^{18} + 81q^{24} - 103q^{26} - 99q^{27} - 91q^{29} + 101q^{31} + 49q^{32} + 41q^{36} + 133q^{39} - 43q^{41} - 184q^{46} - 19q^{47} - 183q^{48} + 200q^{50} + 232q^{52} - 295q^{54} + 209q^{58} + 41q^{62} - 224q^{64} + 253q^{69} + 77q^{71} + 393q^{72} - 283q^{73} - 275q^{75} - 375q^{78} + 418q^{81} - 247q^{82} - 227q^{87} + 161q^{92} - 203q^{93} + 353q^{94} + 616q^{96} +$$

$$392q^{98} - 175q^{100} + O[q]^{101}$$

$$\begin{aligned} \frac{4}{\sqrt{23}} \times \Theta_{L_{a_1}, P_2} = & q^3 - 3q^4 + 5q^6 - 7q^9 + 9q^{13} - 11q^{18} + 13q^{24} - 3q^{26} + 9q^{27} - 15q^{29} - 15q^{31} + \\ & 21q^{32} - 27q^{36} + 17q^{39} + 33q^{41} - 39q^{47} - 19q^{48} + 45q^{54} + 21q^{58} - 51q^{62} - 23q^{69} + 57q^{71} + \\ & 5q^{72} - 15q^{73} + 25q^{75} - 35q^{78} - 38q^{81} + 45q^{82} - 55q^{87} + 69q^{92} + 65q^{93} - 27q^{94} - 75q^{100} + O[q]^{101} \end{aligned}$$

$$\begin{aligned} \Theta_{L_{a_2}} = & 1 + 2q^2 + 2q^3 + 2q^4 + 2q^6 + 2q^8 + 2q^9 + 4q^{12} + 2q^{13} + 4q^{16} + 4q^{18} + 6q^{24} + \\ & 2q^{26} + 2q^{27} + 2q^{29} + 2q^{31} + 4q^{32} + 6q^{36} + 2q^{39} + 2q^{41} + 2q^{46} + 2q^{47} + 6q^{48} + 2q^{50} + 4q^{52} + \\ & 6q^{54} + 2q^{58} + 2q^{62} + 4q^{64} + 2q^{69} + 2q^{71} + 8q^{72} + 2q^{73} + 2q^{75} + 6q^{78} + 4q^{81} + 2q^{82} + 2q^{87} + \\ & 2q^{92} + 2q^{93} + 2q^{94} + 8q^{96} + 2q^{98} + 2q^{100} + O[q]^{101} \end{aligned}$$

$$\begin{aligned} 2 \times \Theta_{L_{a_2}, P_1} = & 8q^2 - 11q^3 - 7q^4 + q^6 + 32q^8 + 13q^9 - 88q^{12} + 29q^{13} - 56q^{16} + 121q^{18} + \\ & 81q^{24} - 103q^{26} - 99q^{27} - 91q^{29} + 101q^{31} + 49q^{32} + 41q^{36} + 133q^{39} - 43q^{41} - 184q^{46} - 19q^{47} - \\ & 183q^{48} + 200q^{50} + 232q^{52} - 295q^{54} + 209q^{58} + 41q^{62} - 224q^{64} + 253q^{69} + 77q^{71} + 393q^{72} - \\ & 283q^{73} - 275q^{75} - 375q^{78} + 418q^{81} - 247q^{82} - 227q^{87} + 161q^{92} - 203q^{93} + 353q^{94} + 616q^{96} + \\ & 392q^{98} - 175q^{100} + O[q]^{101} \end{aligned}$$

$$\begin{aligned} \frac{4}{\sqrt{23}} \times \Theta_{L_{a_2}, P_2} = & -q^3 + 3q^4 - 5q^6 + 7q^9 - 9q^{13} + 11q^{18} - 13q^{24} + 3q^{26} - 9q^{27} + 15q^{29} + 15q^{31} - \\ & 21q^{32} + 27q^{36} - 17q^{39} - 33q^{41} + 39q^{47} + 19q^{48} - 45q^{54} - 21q^{58} + 51q^{62} + 23q^{69} - 57q^{71} - \\ & 5q^{72} + 15q^{73} - 25q^{75} + 35q^{78} + 38q^{81} - 45q^{82} + 55q^{87} - 69q^{92} - 65q^{93} + 27q^{94} + 75q^{100} + O[q]^{101} \end{aligned}$$

We calculate the Hecke character of weight 3 and modulus (1), i.e, we calculate  $\Psi_{K,\Lambda} = \sum_{m \geq 1} a(m)q^m$ , where  $\Lambda = (1)$  and  $k = 3$ . When  $A$  of norm  $m$  is a nonprincipal ideal,  $A^3$  is a principal ideal. Then we set  $\phi(A)^3 = \phi(A^3)$ . For example,  $(2, -1/2 + \sqrt{-23}/2)^3 = (-3/2 - \sqrt{-23}/2)$ . Because of

$$\phi\left(\left(\frac{-3 - \sqrt{-23}}{2}\right)\right) = \left(\frac{-3 - \sqrt{-23}}{2}\right)^2 = \frac{-7 + 3\sqrt{-23}}{2},$$

$\phi((2, -1/2 + \sqrt{-23}/2))$  is one of the roots of

$$(11) \quad x^3 - \left(\frac{-7 + 3\sqrt{-23}}{2}\right) = 0.$$

We denote by  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  the roots of equation (11), namely,  $\alpha_1 \sim -1.86272 + 0.728188i$ ,  $\alpha_2 \sim 0.300733 - 1.97726i$  and  $\alpha_3 \sim 1.56199 + 1.24907i$ , respectively. Then,  $\phi((2, -1/2 + \sqrt{-23}/2))$  is one of  $\alpha_1$ ,  $\alpha_2$  or  $\alpha_3$ . (Actually there are three different Hecke characters in this case.) First let us set  $\phi((2, -1/2 + \sqrt{-23}/2)) = \alpha_1$ . By the equation  $(2, -1/2 + \sqrt{-23}/2) \times (2, 1/2 +$

$$\sqrt{-23}/2 = (2),$$

$$\phi\left(\left(2, \frac{-1 + \sqrt{-23}}{2}\right)\right) \times \phi\left(\left(2, \frac{1 + \sqrt{-23}}{2}\right)\right) = \phi((2)).$$

We get

$$\alpha_1 \times \phi\left(\left(2, \frac{1 + \sqrt{-23}}{2}\right)\right) = 4,$$

hence,  $\phi((2, 1/2 + \sqrt{-23}/2)) = 4/\alpha_1$ . So,

$$a(2) = \phi\left(\left(2, \frac{-1 + \sqrt{-23}}{2}\right)\right) + \phi\left(\left(2, \frac{1 + \sqrt{-23}}{2}\right)\right) = \alpha_1 + 4/\alpha_1.$$

By the equation  $(2, -1/2 + \sqrt{-23}/2) \times (3, 1/2 - \sqrt{-23}/2) = (1/2 - \sqrt{-23}/2)$ ,

$$\phi\left(\left(2, \frac{-1 + \sqrt{-23}}{2}\right)\right) \times \phi\left(\left(3, \frac{1 - \sqrt{-23}}{2}\right)\right) = \phi\left(\left(\frac{1 - \sqrt{-23}}{2}\right)\right).$$

We get

$$\alpha_1 \times \phi\left(\left(3, \frac{1 - \sqrt{-23}}{2}\right)\right) = \left(\frac{1 - \sqrt{-23}}{2}\right)^2 = \frac{-11 - \sqrt{-23}}{2},$$

hence,  $\phi((3, 1/2 - \sqrt{-23}/2)) = (-11 - \sqrt{-23})/2 \times 1/\alpha_1$ . Similarly,  $\phi((3, -1/2 - \sqrt{-23}/2)) = (-11 + \sqrt{-23})/2 \times \alpha_1/(\alpha_1^2 + 4)$ . So,

$$\begin{aligned} a(3) &= \phi\left(\left(3, \frac{1 - \sqrt{-23}}{2}\right)\right) + \phi\left(\left(3, \frac{-1 - \sqrt{-23}}{2}\right)\right) \\ &= \frac{-11 - \sqrt{-23}}{2} \times \frac{1}{\alpha_1} + \frac{-11 + \sqrt{-23}}{2} \times \frac{\alpha_1}{\alpha_1^2 + 4}. \end{aligned}$$

We recall  $\alpha_1 \sim -1.86272 + 0.728188i$ . Then, we obtain

$$\Psi_{K,\Lambda}^{(1)} = q - 3.72545q^2 + 4.24943q^3 + \dots$$

Actually, it is possible to continue this calculation, but we need the information on the basis of all the ideals, which is rather complicated. So, we determine the Hecke eigenforms  $\Psi_{K,\Lambda}^{(i)}$  by a different method. By computer calculation (using ‘‘Sage’’ [17]), we know that the space of the modular forms

of weight 3 where  $\Psi_{K,\Lambda}$  belongs is of dimension 3. We can calculate the basis of this modular form explicitly, and the three basis elements are of the form:

$$\begin{aligned} q + 4q^4 - 11q^6 - 7q^8 + 9q^9 + \cdots, \\ q^2 - 5q^4 + 7q^6 + 4q^8 - 8q^9 + \cdots, \\ q^3 - 3q^4 + 5q^6 - 7q^9 + \cdots. \end{aligned}$$

On the other hand, because of Theorems 3.1 and 3.2,  $\Theta_{L_o, P_1}$ ,  $\Theta_{L_{a_1}, P_1}$  and  $\Theta_{L_{a_2}, P_2}$  are in the same space of Hecke eigenforms  $\Psi_{K,\Lambda}^{(i)}$ . Therefore, comparing the first three coefficients of the following equation:

$$\Psi_{K,\Lambda}^{(1)}(q) = \frac{1}{2}\Theta_{L_o, P}(q) + a2\Theta_{L_{a_1}, P}(q) + b\frac{4}{\sqrt{23}}\Theta_{L_{a_2}, P}(q),$$

we can find numbers  $a$  and  $b$  as follows:

$$(a, b) = \begin{cases} (A_1, B_2), \\ (A_2, B_1), \\ (A_3, B_3), \end{cases}$$

where  $A_1$ ,  $A_2$  and  $A_3$  are the elements defined by

$$\begin{aligned} \{x \mid 512x^3 - 96x + 7 = 0\} \\ = \{A_1 = -0.465681, A_2 = 0.0751832, A_3 = 0.390498\}, \end{aligned}$$

respectively, and  $B_1$ ,  $B_2$  and  $B_3$  are the elements defined by

$$\begin{aligned} \{x \mid 512x^3 - 2208x + 1587 = 0\} \\ = \{B_1 = -2.37065, B_2 = 0.873067, B_3 = 1.49759\}, \end{aligned}$$

respectively.

In this way, we can calculate the Hecke eigenforms  $\Psi_{K,\Lambda}^{(i)}$ . Namely,  
 $\Psi_{K,\Lambda}^{(1)} = q - 3.72545q^2 + 4.24943q^3 + 9.87897q^4 - 15.831q^6 - 21.9018q^8 + 9.05761q^9 + 41.9799q^{12} - 21.3624q^{13} + 42.0781q^{16} - 33.7437q^{18} - 23q^{23} - 93.07q^{24} + 25q^{25} + 79.5844q^{26} + 0.244826q^{27} + 55.473q^{29} - 33.9378q^{31} - 69.1528q^{32} + 89.4799q^{36} - 90.7777q^{39} - 8.78692q^{41} + 85.6853q^{46} + 42.8975q^{47} + 178.808q^{48} + 49q^{49} - 93.1362q^{50} + O[q]^{51}.$

$$\begin{aligned} \Psi_{K,\Lambda}^{(2)} = q + 0.601466q^2 + 1.54364q^3 - 3.63824q^4 + 0.928445q^6 - 4.59414q^8 - 6.61718q^9 - 5.61612q^{12} + 23.5162q^{13} + 11.7897q^{16} - 3.98001q^{18} - 23q^{23} - 7.09168q^{24} + 25q^{25} + 14.1442q^{26} - \end{aligned}$$



$$24.1073q^{27} - 42.4015q^{29} - 27.9663q^{31} + 25.4677q^{32} + 24.0749q^{36} + 36.3005q^{39} + 74.9986q^{41} - 13.8337q^{46} - 93.8839q^{47} + 18.1991q^{48} + 49q^{49} + 15.0366q^{50} + O[q]^{51}.$$

$$\begin{aligned} \Psi_{K,\Lambda}^{(3)} = & q + 3.12398q^2 - 5.79306q^3 + 5.75927q^4 - 18.0974q^6 + 5.49593q^8 + 24.5596q^9 - \\ & 33.3638q^{12} - 2.15383q^{13} - 5.86788q^{16} + 76.7237q^{18} - 23q^{23} - 31.8383q^{24} + 25q^{25} - 6.72853q^{26} - \\ & 90.1376q^{27} - 13.0715q^{29} + 61.9041q^{31} - 40.3149q^{32} + 141.445q^{36} + 12.4773q^{39} - 66.2117q^{41} - \\ & 71.8516q^{46} + 50.9864q^{47} + 33.993q^{48} + 49q^{49} + 78.0996q^{50} + O[q]^{51}. \end{aligned}$$

The coefficients  $a(m)$  for this case are far from integers. In fact they are not elements in a cyclotomic number field in general. So, it seems difficult to use the Hecke eigenforms obtained this way to apply for the case of the class number 3 or more in general. Some new additional ideas will be needed to treat the case of  $d = 23$  or more generally the cases of class numbers  $h \geq 3$ . We have included the presentation of the results (although they are not conclusive) for  $d = 23$ , hoping that it might help the reader for the future study on this topic.

**Remark 6.1.** We remark that the coefficients of  $\Psi_{K,\Lambda}^{(i)}$  in above calculator results are not exact values but approximate values.

Table 4: Integral ideals of small norm of  $d = 23$

$N(A)$	$A$ : ideal	$N(A)$	$A$ : ideal
1	(1)	6	$(1/2 - \sqrt{-23}/2)$ $(6, 5/2 + \sqrt{-23}/2)$ $(6, 7/2 + \sqrt{-23}/2)$ $(1/2 + \sqrt{-23}/2)$
2	$(2, -1/2 + \sqrt{-23}/2)$ $(2, 1/2 + \sqrt{-23}/2)$	7	—
3	$(3, 1/2 - \sqrt{-23}/2)$ $(3, -1/2 - \sqrt{-23}/2)$	8	$(-3/2 - \sqrt{-23}/2)$ $(4, -1 + \sqrt{-23})$ $(4, 1 + \sqrt{-23})$ $(-3/2 + \sqrt{-23}/2)$
4	$(4, 3/2 + \sqrt{-23}/2)$ (2) $(4, 5/2 + \sqrt{-23}/2)$	9	$(9, 11/2 + \sqrt{-23}/2)$ (3) $(9, 7/2 + \sqrt{-23}/2)$
5	—	10	—

## 7 Concluding Remarks

- (1) In this paper, we use the mathematics software “Sage” [17]. In particular, The results in Tables 1 and 4 are compute by “Sage” using the command “K.ideals\_of\_bdd\_norm()”. We remark that this command do not always give a  $\mathbb{Z}$ -basis of ideal. We must make sure the command “(ideal).basis()”.
- (2) In Appendix C, we list theta series of lattices obtained from  $\mathbb{Q}(\sqrt{-5})$ . The other cases are listed in one of the author’s website [12].
- (3) In the previous paper [3], we studied the spherical designs in the nonempty shells of the  $\mathbb{Z}^2$ -lattice and  $A_2$ -lattice. The results state that any shells in the  $\mathbb{Z}^2$ -lattice (resp.  $A_2$ -lattice) are spherical 2-design (resp. 4-design). However, the nonempty shells in the  $\mathbb{Z}^2$ -lattice (resp.  $A_2$ -lattice) are not spherical 4-design (resp. 6-design). It is interesting to note that no spherical 6-design among the nonempty shells of any Euclidean lattice of 2-dimension is known. It is an interesting open problem to prove or disprove whether these exists any 6-design which is a shell of a Euclidean lattice of 2-dimension.

Responding to the authors’ request, Junichi Shigezumi performed computer calculations to determine whether there are spherical  $t$ -designs for bigger  $t$ , in the 2- and 3-dimensional cases. His calculation shows that among the nonempty shells of integral lattices in 2-dimension (with relatively small discriminant and small norms), there are only 4-designs. That is, no 6-designs were found. (So far, all examples of such 4-designs are the union of vertices of regular 6-gons, although they are the nonempty shells of many different lattices). In the 3-dimensional case, all examples obtained are only 2-designs. No 4-designs which are shells of a lattice were found. It is an interesting open problem whether this is true in general for the dimensions 2 and 3. Moreover, it is interesting to note that no spherical 12-design among the nonempty shells of any Euclidean lattice (of any dimension) is known. It is also an interesting open problem to prove or disprove whether these exists any 12-design which is a shell of a Euclidean lattice.

Finally, we state the following conjecture for the 2-dimensional lattices.

**Conjecture 7.1.** *Let  $L$  be the Euclidean lattice of 2-dimension, whose quadratic form is  $ax^2 + bxy + cy^2$ .*

- (i) Assume that  $b^2 - 4ac = (\text{Integer})^2 \times (-3)$ . Then, all the nonempty shells of  $L$  are not spherical 6-designs and some of the nonempty shells of  $L$  are spherical 4-designs. Moreover, if all the nonempty shells of  $L$  are spherical 4-designs then  $b^2 - 4ac = -3$ , that is,  $A_2$ -lattice.
- (ii) Assume that  $b^2 - 4ac = (\text{Integer})^2 \times (-4)$ . Then, all the nonempty shells of  $L$  are not spherical 4-designs and some of the nonempty shells of  $L$  are spherical 2-designs. Moreover, if all the nonempty shells of  $L$  are spherical 2-designs then  $b^2 - 4ac = -4$ , that is,  $\mathbb{Z}^2$ -lattice.
- (iii) Otherwise, all the nonempty shells of  $L$  are not spherical 2-designs.

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## A The case of $|\text{Cl}_K| = 1$

Table 5:  $|\text{Cl}_K| = 1$

$-d$	$-d \pmod{4}$	$d_K$	$L_{\mathfrak{o}}$
-1	3	$-2^2$	$[1, \sqrt{-1}]$
-2	2	$-2^3$	$[1, \sqrt{-2}]$
-3	1	-3	$[1, (1 + \sqrt{-3})/2]$
-7	1	-7	$[1, (1 + \sqrt{-7})/2]$
-11	1	-11	$[1, (1 + \sqrt{-11})/2]$
-19	1	-19	$[1, (1 + \sqrt{-19})/2]$
-43	1	-43	$[1, (1 + \sqrt{-43})/2]$
-67	1	-67	$[1, (1 + \sqrt{-67})/2]$
-163	1	-163	$[1, (1 + \sqrt{-163})/2]$

## B The case of $|\text{Cl}_K| = 2$

Table 6:  $|\text{Cl}_K| = 2$

$-d$	$-d \pmod{4}$	$d_K$	$L_{\mathfrak{o}}$	$L_{\mathfrak{a}}$
-5	3	$-2^2 \times 5$	$[1, \sqrt{-5}]$	$[2, 1 + \sqrt{-5}]$
-6	2	$-2^3 \times 3$	$[1, \sqrt{-6}]$	$[2, \sqrt{-6}]$
-10	2	$-2^3 \times 5$	$[1, \sqrt{-10}]$	$[2, \sqrt{-10}]$
-13	3	$-2^2 \times 13$	$[1, \sqrt{-13}]$	$[2, 1 + \sqrt{-13}]$
-15	1	$-3 \times 5$	$[1, (1 + \sqrt{-15})/2]$	$[2, (1 + \sqrt{-15})/2]$
-22	2	$-2^3 \times 11$	$[1, \sqrt{-22}]$	$[2, \sqrt{-22}]$
-35	1	$-5 \times 7$	$[1, (1 + \sqrt{-35})/2]$	$[3, (1 + \sqrt{-35})/2]$
-37	3	$-2^2 \times 37$	$[1, \sqrt{-37}]$	$[2, 1 + \sqrt{-37}]$
-51	1	$-3 \times 17$	$[1, (1 + \sqrt{-51})/2]$	$[3, (3 + \sqrt{-51})/2]$
-58	2	$-2^3 \times 29$	$[1, \sqrt{-58}]$	$[2, \sqrt{-58}]$
-91	1	$-7 \times 13$	$[1, (1 + \sqrt{-91})/2]$	$[5, (3 + \sqrt{-91})/2]$
-115	1	$-5 \times 23$	$[1, (1 + \sqrt{-115})/2]$	$[5, (5 + \sqrt{-115})/2]$
-123	1	$-3 \times 41$	$[1, (1 + \sqrt{-123})/2]$	$[3, (3 + \sqrt{-123})/2]$
-187	1	$-11 \times 17$	$[1, (1 + \sqrt{-187})/2]$	$[7, (3 + \sqrt{-187})/2]$
-235	1	$-5 \times 47$	$[1, (1 + \sqrt{-235})/2]$	$[5, (5 + \sqrt{-235})/2]$
-267	1	$-3 \times 89$	$[1, (1 + \sqrt{-267})/2]$	$[3, (3 + \sqrt{-267})/2]$
-403	1	$-13 \times 31$	$[1, (1 + \sqrt{-403})/2]$	$[11, (9 + \sqrt{-403})/2]$
-427	1	$-7 \times 61$	$[1, (1 + \sqrt{-427})/2]$	$[7, (7 + \sqrt{-427})/2]$

## C Theta series of $L_0$ and $L_a$ of $\mathbb{Q}(\sqrt{-5})$

$$\begin{aligned}\Theta_{L_0} = & 1 + 2q + 2q^4 + 2q^5 + 4q^6 + 6q^9 + 4q^{14} + 2q^{16} + 2q^{20} + 8q^{21} + 4q^{24} + 2q^{25} + 4q^{29} + 4q^{30} + \\ & 6q^{36} + 4q^{41} + 6q^{45} + 4q^{46} + 6q^{49} + 8q^{54} + 4q^{56} + 4q^{61} + 2q^{64} + 8q^{69} + 4q^{70} + 2q^{80} + 10q^{81} + \\ & 8q^{84} + 4q^{86} + 4q^{89} + 4q^{94} + 4q^{96} + 2q^{100} + 4q^{101} + 8q^{105} + 4q^{109} + 4q^{116} + 4q^{120} + 2q^{121} + \\ & 2q^{125} + 12q^{126} + 8q^{129} + 4q^{134} + 8q^{141} + 6q^{144} + 4q^{145} + 4q^{149} + 4q^{150} + 8q^{161} + 4q^{164} + \\ & 4q^{166} + 2q^{169} + 8q^{174} + 6q^{180} + 4q^{181} + 4q^{184} + 16q^{189} + 6q^{196} + 8q^{201} + 4q^{205} + 4q^{206} + 4q^{214} + \\ & 8q^{216} + 4q^{224} + 6q^{225} + 4q^{229} + 4q^{230} + 4q^{241} + 4q^{244} + 6q^{245} + 8q^{246} + 8q^{249} + 4q^{254} + 2q^{256} + \\ & 12q^{261} + 4q^{269} + 8q^{270} + 8q^{276} + 4q^{280} + 4q^{281} + 2q^{289} + 12q^{294} + 8q^{301} + 4q^{305} + 8q^{309} + \\ & 2q^{320} + 8q^{321} + 10q^{324} + 4q^{326} + 8q^{329} + 4q^{334} + 8q^{336} + 4q^{344} + 8q^{345} + 4q^{349} + 4q^{350} + 4q^{356} + \\ & 2q^{361} + 8q^{366} + 12q^{369} + 4q^{376} + 8q^{381} + 4q^{384} + 4q^{389} + 2q^{400} + 4q^{401} + 4q^{404} + 10q^{405} + \\ & 8q^{406} + 4q^{409} + 12q^{414} + 8q^{420} + 4q^{421} + 4q^{430} + 4q^{436} + 18q^{441} + 4q^{445} + 4q^{446} + 4q^{449} + \\ & 4q^{454} + 4q^{461} + 4q^{464} + 8q^{469} + 4q^{470} + 4q^{480} + 2q^{484} + 12q^{486} + 8q^{489} + 2q^{500} + O[q]^{501}\end{aligned}$$

$$\begin{aligned}\Theta_{L_a} = & 1 + 2q^2 + 4q^3 + 4q^7 + 2q^8 + 2q^{10} + 4q^{12} + 4q^{15} + 6q^{18} + 4q^{23} + 8q^{27} + 4q^{28} + 2q^{32} + \\ & 4q^{35} + 2q^{40} + 8q^{42} + 4q^{43} + 4q^{47} + 4q^{48} + 2q^{50} + 4q^{58} + 4q^{60} + 12q^{63} + 4q^{67} + 6q^{72} + 4q^{75} + \\ & 4q^{82} + 4q^{83} + 8q^{87} + 6q^{90} + 4q^{92} + 6q^{98} + 4q^{103} + 4q^{107} + 8q^{108} + 4q^{112} + 4q^{115} + 4q^{122} + \\ & 8q^{123} + 4q^{127} + 2q^{128} + 8q^{135} + 8q^{138} + 4q^{140} + 12q^{147} + 2q^{160} + 10q^{162} + 4q^{163} + 4q^{167} + \\ & 8q^{168} + 4q^{172} + 4q^{175} + 4q^{178} + 8q^{183} + 4q^{188} + 4q^{192} + 2q^{200} + 4q^{202} + 8q^{203} + 12q^{207} + 8q^{210} + \\ & 4q^{215} + 4q^{218} + 4q^{223} + 4q^{227} + 4q^{232} + 4q^{235} + 4q^{240} + 2q^{242} + 12q^{243} + 2q^{250} + 12q^{252} + \\ & 8q^{258} + 4q^{263} + 8q^{267} + 4q^{268} + 8q^{282} + 4q^{283} + 8q^{287} + 6q^{288} + 4q^{290} + 4q^{298} + 4q^{300} + 8q^{303} + \\ & 4q^{307} + 12q^{315} + 8q^{322} + 8q^{327} + 4q^{328} + 4q^{332} + 4q^{335} + 2q^{338} + 8q^{343} + 4q^{347} + 8q^{348} + 6q^{360} + \\ & 4q^{362} + 4q^{363} + 4q^{367} + 4q^{368} + 4q^{375} + 16q^{378} + 4q^{383} + 12q^{387} + 6q^{392} + 8q^{402} + 4q^{410} + \\ & 4q^{412} + 4q^{415} + 12q^{423} + 8q^{427} + 4q^{428} + 8q^{432} + 8q^{435} + 4q^{443} + 8q^{447} + 4q^{448} + 6q^{450} + 4q^{458} + \\ & 4q^{460} + 4q^{463} + 4q^{467} + 4q^{482} + 16q^{483} + 4q^{487} + 4q^{488} + 6q^{490} + 8q^{492} + 8q^{498} + O[q]^{501}\end{aligned}$$

$$\begin{aligned}\Theta_{L_{a,P}} = & q + 4q^4 - 5q^5 - 8q^6 + 7q^9 + 8q^{14} + 16q^{16} - 20q^{20} - 16q^{21} - 32q^{24} + 25q^{25} - \\ & 22q^{29} + 40q^{30} + 28q^{36} + 62q^{41} - 35q^{45} - 88q^{46} - 33q^{49} + 16q^{54} + 32q^{56} - 58q^{61} + 64q^{64} + \\ & 176q^{69} - 40q^{70} - 80q^{80} - 95q^{81} - 64q^{84} + 152q^{86} - 142q^{89} + 8q^{94} - 128q^{96} + 100q^{100} + \\ & 122q^{101} + 80q^{105} + 38q^{109} - 88q^{116} + 160q^{120} + 121q^{121} - 125q^{125} + 56q^{126} - 304q^{129} - \\ & 232q^{134} - 16q^{141} + 112q^{144} + 110q^{145} + 278q^{149} - 200q^{150} - 176q^{161} + 248q^{164} + 152q^{166} + \\ & 169q^{169} + 176q^{174} - 140q^{180} - 358q^{181} - 352q^{184} + 32q^{189} - 132q^{196} + 464q^{201} - 310q^{205} - \\ & 88q^{206} + 248q^{214} + 64q^{216} + 128q^{224} + 175q^{225} - 262q^{229} + 440q^{230} + 302q^{241} - 232q^{244} + \\ & 165q^{245} - 496q^{246} - 304q^{249} - 472q^{254} + 256q^{256} - 154q^{261} + 38q^{269} - 80q^{270} + 704q^{276} - \\ & 160q^{280} - 418q^{281} + 289q^{289} + 264q^{294} + 304q^{301} + 290q^{305} + 176q^{309} - 320q^{320} - 496q^{321} - \\ & 380q^{324} - 328q^{326} + 16q^{329} + 488q^{334} - 256q^{336} + 608q^{344} - 880q^{345} - 22q^{349} + 200q^{350} - \\ & 568q^{356} + 361q^{361} + 464q^{366} + 434q^{369} + 32q^{376} + 944q^{381} - 512q^{384} - 202q^{389} + 400q^{400} - \\ & 478q^{401} + 488q^{404} + 475q^{405} - 176q^{406} - 802q^{409} - 616q^{414} + 320q^{420} - 778q^{421} - \\ & 760q^{430} + 152q^{436} - 231q^{441} + 710q^{445} + 872q^{446} + 398q^{449} - 712q^{454} + 842q^{461} - \\ & 352q^{464} - 464q^{469} - 40q^{470} + 640q^{480} + 484q^{484} + 616q^{486} + 656q^{489} - 500q^{500} + O[q]^{501}\end{aligned}$$

$$\begin{aligned}\Theta_{L_{a,F}} = & 2q^2 - 4q^3 + 4q^7 + 8q^8 - 10q^{10} - 16q^{12} + 20q^{15} + 14q^{18} - 44q^{23} + 8q^{27} + 16q^{28} + \\ & 32q^{32} - 20q^{35} - 40q^{40} - 32q^{42} + 76q^{43} + 4q^{47} - 64q^{48} + 50q^{50} - 44q^{58} + 80q^{60} + 28q^{63} - \\ & 116q^{67} + 56q^{72} - 100q^{75} + 124q^{82} + 76q^{83} + 88q^{87} - 70q^{90} - 176q^{92} - 66q^{98} - 44q^{103} + \\ & 124q^{107} + 32q^{108} + 64q^{112} + 220q^{115} - 116q^{122} - 248q^{123} - 236q^{127} + 128q^{128} - 40q^{135} + \\ & 352q^{138} - 80q^{140} + 132q^{147} - 160q^{160} - 190q^{162} - 164q^{163} + 244q^{167} - 128q^{168} + 304q^{172} +\end{aligned}$$

$$\begin{aligned}
& 100q^{175} - 284q^{178} + 232q^{183} + 16q^{188} - 256q^{192} + 200q^{200} + 244q^{202} - 88q^{203} - 308q^{207} + \\
& 160q^{210} - 380q^{215} + 76q^{218} + 436q^{223} - 356q^{227} - 176q^{232} - 20q^{235} + 320q^{240} + 242q^{242} + \\
& 308q^{243} - 250q^{250} + 112q^{252} - 608q^{258} - 284q^{263} + 568q^{267} - 464q^{268} - 32q^{282} + 316q^{283} + \\
& 248q^{287} + 224q^{288} + 220q^{290} + 556q^{298} - 400q^{300} - 488q^{303} - 596q^{307} - 140q^{315} - 352q^{322} - \\
& 152q^{327} + 496q^{328} + 304q^{332} + 580q^{335} + 338q^{338} - 328q^{343} - 116q^{347} + 352q^{348} - \\
& 280q^{360} - 716q^{362} - 484q^{363} + 724q^{367} - 704q^{368} + 500q^{375} + 64q^{378} - 44q^{383} + 532q^{387} - \\
& 264q^{392} + 928q^{402} - 620q^{410} - 176q^{412} - 380q^{415} + 28q^{423} - 232q^{427} + 496q^{428} + 128q^{432} - \\
& 440q^{435} + 796q^{443} - 1112q^{447} + 256q^{448} + 350q^{450} - 524q^{458} + 880q^{460} - 764q^{463} + \\
& 124q^{467} + 604q^{482} + 704q^{483} + 484q^{487} - 464q^{488} + 330q^{490} - 992q^{492} - 608q^{498} + O[q]^{501}
\end{aligned}$$

$$\begin{aligned}
\Psi_{K,\Lambda}^{(1)}(z) = & q + 2q^2 - 4q^3 + 4q^4 - 5q^5 - 8q^6 + 4q^7 + 8q^8 + 7q^9 - 10q^{10} - 16q^{12} + 8q^{14} + 20q^{15} + \\
& 16q^{16} + 14q^{18} - 20q^{20} - 16q^{21} - 44q^{23} - 32q^{24} + 25q^{25} + 8q^{27} + 16q^{28} - 22q^{29} + 40q^{30} + \\
& 32q^{32} - 20q^{35} + 28q^{36} - 40q^{40} + 62q^{41} - 32q^{42} + 76q^{43} - 35q^{45} - 88q^{46} + 4q^{47} - 64q^{48} - 33q^{49} + \\
& 50q^{50} + 16q^{54} + 32q^{56} - 44q^{58} + 80q^{60} - 58q^{61} + 28q^{63} + 64q^{64} - 116q^{67} + 176q^{69} - 40q^{70} + \\
& 56q^{72} - 100q^{75} - 80q^{80} - 95q^{81} + 124q^{82} + 76q^{83} - 64q^{84} + 152q^{86} + 88q^{87} - 142q^{89} - 70q^{90} - \\
& 176q^{92} + 8q^{94} - 128q^{96} - 66q^{98} + 100q^{100} + 122q^{101} - 44q^{103} + 80q^{105} + 124q^{107} + 32q^{108} + \\
& 38q^{109} + 64q^{112} + 220q^{115} - 88q^{116} + 160q^{120} + 121q^{121} - 116q^{122} - 248q^{123} - 125q^{125} + \\
& 56q^{126} - 236q^{127} + 128q^{128} - 304q^{129} - 232q^{134} - 40q^{135} + 352q^{138} - 80q^{140} - 16q^{141} + \\
& 112q^{144} + 110q^{145} + 132q^{147} + 278q^{149} - 200q^{150} - 160q^{160} - 176q^{161} - 190q^{162} - 164q^{163} + \\
& 248q^{164} + 152q^{166} + 244q^{167} - 128q^{168} + 169q^{169} + 304q^{172} + 176q^{174} + 100q^{175} - 284q^{178} - \\
& 140q^{180} - 358q^{181} + 232q^{183} - 352q^{184} + 16q^{188} + 32q^{189} - 256q^{192} - 132q^{196} + 200q^{200} + \\
& 464q^{201} + 244q^{202} - 88q^{203} - 310q^{205} - 88q^{206} - 308q^{207} + 160q^{210} + 248q^{214} - 380q^{215} + \\
& 64q^{216} + 76q^{218} + 436q^{223} + 128q^{224} + 175q^{225} - 356q^{227} - 262q^{229} + 440q^{230} - 176q^{232} - \\
& 20q^{235} + 320q^{240} + 302q^{241} + 242q^{242} + 308q^{243} - 232q^{244} + 165q^{245} - 496q^{246} - 304q^{249} - \\
& 250q^{250} + 112q^{252} - 472q^{254} + 256q^{256} - 608q^{258} - 154q^{261} - 284q^{263} + 568q^{267} - 464q^{268} + \\
& 38q^{269} - 80q^{270} + 704q^{276} - 160q^{280} - 418q^{281} - 32q^{282} + 316q^{283} + 248q^{287} + 224q^{288} + \\
& 289q^{289} + 220q^{290} + 264q^{294} + 556q^{298} - 400q^{300} + 304q^{301} - 488q^{303} + 290q^{305} - 596q^{307} + \\
& 176q^{309} - 140q^{315} - 320q^{320} - 496q^{321} - 352q^{322} - 380q^{324} - 328q^{326} - 152q^{327} + 496q^{328} + \\
& 16q^{329} + 304q^{332} + 488q^{334} + 580q^{335} - 256q^{336} + 338q^{338} - 328q^{343} + 608q^{344} - 880q^{345} - \\
& 116q^{347} + 352q^{348} - 22q^{349} + 200q^{350} - 568q^{356} - 280q^{360} + 361q^{361} - 716q^{362} - 484q^{363} + \\
& 464q^{366} + 724q^{367} - 704q^{368} + 434q^{369} + 500q^{375} + 32q^{376} + 64q^{378} + 944q^{381} - 44q^{383} - \\
& 512q^{384} + 532q^{387} - 202q^{389} - 264q^{392} + 400q^{400} - 478q^{401} + 928q^{402} + 488q^{404} + 475q^{405} - \\
& 176q^{406} - 802q^{409} - 620q^{410} - 176q^{412} - 616q^{414} - 380q^{415} + 320q^{420} - 778q^{421} + 28q^{423} - \\
& 232q^{427} + 496q^{428} - 760q^{430} + 128q^{432} - 440q^{435} + 152q^{436} - 231q^{441} + 796q^{443} + 710q^{445} + \\
& 872q^{446} - 1112q^{447} + 256q^{448} + 398q^{449} + 350q^{450} - 712q^{454} - 524q^{458} + 880q^{460} + 842q^{461} - \\
& 764q^{463} - 352q^{464} + 124q^{467} - 464q^{469} - 40q^{470} + 640q^{480} + 604q^{482} + 704q^{483} + 484q^{484} + \\
& 616q^{486} + 484q^{487} - 464q^{488} + 656q^{489} + 330q^{490} - 992q^{492} - 608q^{498} - 500q^{500} + O[q]^{501}
\end{aligned}$$

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